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# Exact spatially inhomogeneous cosmologies

J Wainwright

Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1

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**Abstract.** The aim of this paper is to give an overview of the derivation and properties of exact solutions of the Einstein field equations which are spatially inhomogeneous with the source assumed to be an irrotational perfect fluid. It is shown that the known such spatially inhomogeneous solutions either admit a group of isometries with 2D orbits or are algebraically special. The solutions are related to a previously given classification scheme which is based on the intrinsic and extrinsic geometry of the hypersurfaces orthogonal to the fluid flow.

## 1. Introduction

A theoretical discussion of many aspects of the early universe (for example galaxy formation, the possibility of primordial black hole formation, the nature of the initial singularity) necessitates the use of spatially inhomogeneous cosmological models (see for example Carr and Hawking 1974, Liang 1976, Barrow 1978, Centrella and Matzner 1979, Barrow 1980). The task is made more difficult by a lack of suitable exact solutions of the Einstein field equations (see for example, the remark on p 510 in Pollock and Caderni (1980), the footnote on p 274 in Barrow (1980), and Collins and Szafron (1979)). For this reason, most work to date has relied on linear perturbation methods (see for example Olson (1976) and references therein).

The purpose of this paper is to survey and classify the known exact solutions of the Einstein field equations (subsequently abbreviated as EFEs), which can possibly be interpreted as spatially inhomogeneous cosmological models. We thus require first of all that the metric be *spatially inhomogeneous* in the sense that†

$A_1$: the orbits of the maximal group of local isometries have dimension $\leqslant 2$.

We stress that this assumption excludes the well known spatially homogeneous solutions with perfect fluid source, both the non-tilted ones (Ellis and MacCallum 1969) and the tilted ones (King and Ellis 1973).

As regards the source term in the EFEs, we will restrict our attention to non-vacuum solutions in which the matter–energy content is idealised to be a perfect fluid. We are thereby excluding certain known spatially inhomogeneous vacuum solutions which are of interest as cosmological models, for example the Gowdy (1971) solutions.

For the purposes of the survey, the only assumption that we will make about the nature of the solutions is that they are *evolving in time*. This leads us to assume that the

† See Goode (1980) for a discussion of different ways of defining spatial inhomogeneity.

perfect fluid has *non-zero (rate of) expansion*†. This assumption excludes a number of spatially inhomogeneous solutions (for example the Winicour (1975) dust solutions and the Wahlquist (1968) perfect fluid solution, both of which admit one space-like and one time-like Killing vector field). This exclusion is reasonable on physical grounds, however, since the natural physical interpretation of perfect fluid solutions with zero expansion is as models of (rotating) distributions of perfect fluid *in equilibrium*.

We now consider the vorticity of the perfect fluid. The only known‡ solution of the EFEs with perfect fluid source which has non-zero expansion and non-zero vorticity is the solution of Demianski and Grishchuck (1972) (which is exact modulo solution of an ordinary differential equation). This solution, however, admits a group of local isometries with three-dimensional orbits and hence does not satisfy $A_1$. To reiterate: *there are no known spatially inhomogeneous exact solutions in which the perfect fluid source has non-zero expansion and non-zero rotation.* Thus, at the present time, a survey of known spatially inhomogeneous solutions of the EFEs, whose source is a perfect fluid with non-zero expansion, is of necessity restricted to solutions in which the vorticity of the fluid is zero. Our second assumption is thus

$A_2$: the source is a perfect fluid with zero vorticity and non-zero expansion.

Any exact solution of the EFEs will inevitably be 'special' in some respect, although not necessarily in the sense of admitting a local group of isometries. In fact there are several known classes of solutions subject to $A_1$ and $A_2$, which admit no Killing vector fields (see §5). These solutions are 'special', however, as regards the algebraic structure of the Weyl tensor. Indeed, it turns out that all known exact solutions of the EFEs subject to $A_1$ and $A_2$ which admit less that two Killing vector fields are *algebraically special*,§ that is, of type II, D, III, N or O in the Petrov classification (see for example Kramer *et al* (1980, pp 58–65) for this terminology). The situation can thus be summed up by stating that all known solutions of the EFEs subject to $A_1$ and $A_2$ satisfy at least one of the following additional restrictions:

$R_1$: the metric admits a group of local isometries with two-dimensional space-like orbits,

$R_2$: the Weyl tensor is algebraically special.

This fact is used to simplify the classification given in this paper.

The earliest spatially inhomogeneous solutions were the Tolman–Bondi spherically symmetric models (see Tolman 1934, Bondi 1947), and the Eardley *et al* (1972) plane-symmetric models, both of which have pressure-free matter as source. Both of these solutions admit a three-parameter group of local isometries with two-dimensional orbits, and hence are members of the class of LRS solutions (Ellis 1967, Stewart and Ellis 1968), which satisfy conditions $R_1$ *and* $R_2$. Indeed the LRS solutions admit two repeated principal null directions, and hence are of Petrov type D (Wainwright 1970).

The LRS solutions have been studied extensively, but even in this special case, the field equations have not been solved in general except in the case when the source is

† We are certainly not asserting that any such solution will provide a reasonable model of the early inhomogeneous universe; at this stage we merely wish to impose a minimal set of restrictions. We should also mention that Ellis *et al* (1978) have recently studied the possibility of a static spherically symmetric universe, in which the rate of expansion is zero, but find that it is difficult to construct plausible cosmological models of this type.

‡ To this and similar statements made subsequently, one should add the disclaimer 'to the best of the author's knowledge', i.e. based on the author's knowledge of the research literature, and a perusal of the survey volume on exact solutions by Kramer *et al* (1980).

§ This does not apply to vacuum solutions. See Kramer *et al* (1980, pp 178–9).

dust. Since the general properties of this class of solutions are well known, we will not consider them further here (see Kramer *et al* (1980) for a survey of explicit exact solutions). In the subclass of solutions defined by $R_1$, there remain the solutions in which the local group of isometries is a two-parameter group. In all known solutions this group is *Abelian*. For this reason, this survey is henceforth restricted to solutions which admit an Abelian $G_2$ (§ 4) and solutions which are algebraically special (§ 5).

Spatially inhomogeneous solutions have invariably been derived either by assuming the existence of a local group of isometries or by assuming that the Weyl tensor is algebraically special. An interesting class of spatially inhomogeneous solutions was found by Szekeres (1975) using neither of these methods. Instead he simply imposed an *ad hoc* assumption on the metric, namely that it had a particular diagonal form, with the components depending on all four coordinates. It was subsequently discovered, somewhat surprisingly, that the Szekeres solutions were in fact of Petrov type D and hence satisfied restriction $R_2$ (Wainwright 1977). It was also discovered that the Szekeres solutions were remarkably simple from a different point of view, namely the intrinsic geometry of the hypersurfaces orthogonal to the fluid flow. In particular *these hypersurfaces are conformally flat* (Berger *et al* 1977, Szafron and Collins 1979). This, and other properties of the Szekeres solutions, led Collins and Szafron (1979) to propose a classification scheme for spatially inhomogeneous cosmologies. Further investigation of known spatially inhomogeneous cosmologies suggested that the so-called Cotton–York tensor, which describes the conformal geometry of the space-like hypersurfaces, should be incorporated into any classification scheme. This was done in a recent paper by the present author (Wainwright 1979); this paper will be subsequently referred to as I. It will be assumed that the reader is familiar with the terminology and classification scheme of this paper. In the present paper we describe the properties of the known non-rotating spatially inhomogeneous solutions in terms of this classification scheme.

In § 2 we give a classification of space–times which admit an Abelian $G_2$, and derive canonical forms for the tensors used in the classification scheme of paper I, namely the spatial Ricci tensor $R^*_{\alpha\beta}$, the spatial Cotton–York tensor $C^*_{\alpha\beta}$ and the rate of shear tensor of the normal congruence $\sigma_{\alpha\beta}$. In § 3, canonical coordinates are constructed for space–times which admit an Abelian $G_2$, and in § 4 the known spatially inhomogeneous solutions of this type are surveyed. Section 5 deals with non-rotating spatially inhomogeneous solutions which are algebraically special.

## 2. Space–times which admit an Abelian $G_2$: classification

In this section we consider space-times which admit an Abelian $G_2$ of local isometries. We will consider additional restrictions which lead to greater simplification of the field equations, namely whether or not the group acts orthogonally transitively (Carter 1969) and whether or not the group admits a hypersurface-orthogonal (HO) Killing vector field (KVF) as a generator. The following theorem enables us to distinguish four mutually exclusive classes of space–times which admit an Abelian $G_2$.

*Theorem 2.1.* Suppose that one of the Killing vector fields $\xi$ of an Abelian $G_2$ with space-like orbits is hypersurface-orthogonal. Then the other Killing vector field $\eta$ can be chosen to be orthogonal to $\xi$. In addition $\eta$ is hypersurface-orthogonal if and only if the group acts orthogonally transitively.

*Proof.* See the Appendix.

The four mutually exclusive classes of space–times are labelled as follows:
A(i): non-orthogonally transitive $G_2$, with no HO KVFS;
A(ii): non-orthogonally transitive $G_2$, with one HO KVF;
B(i): orthogonally transitive $G_2$, with no HO KVFS;
B(ii): orthogonally transitive $G_2$, with two mutually orthogonal HO KVFS.

*Remarks.* (i) We have not been able to exclude the possibility that there are solutions in class A(ii) having two HO KVFS which are *not* mutually orthogonal.

(ii) It is conceivable that solutions of class A(i) may admit two mutually orthogonal KVFS; however, neither of them could be HO.

We now select a unit time-like vector field $u$ which is orthogonal to the group orbits and is invariant under the group. Any such vector field is hypersurface-orthogonal for the following reason. If local coordinates $t$, $x$, $y$, $z$ are introduced such that the KVFS of the $G_2$ are given by

$$\xi = \partial/\partial y, \qquad \eta = \partial/\partial z,$$

then the one-form corresponding to $u$ will have the form

$$u_i \, dx^i = \alpha \, dt + \beta \, dx,$$

where $\alpha$, $\beta$ are independent of $y$ and $z$. This implies that $u_{[i,j}u_{k]} = 0$. We will regard the chosen vector field $u$ as a preferred vector field, which will subsequently be identified with the fluid velocity field.

We now give the canonical forms for the rank-two tensors $R^*_{\alpha\beta}$, $C^*_{\alpha\beta}$ and $\sigma_{\alpha\beta}$, which determine the intrinsic and extrinsic geometry of the space-like hypersurfaces orthogonal to $u$. As in the Appendix, we can choose an orthonormal frame with $e_0 = u$ and $e_1$ orthogonal to the orbits, and which is invariant under the group. This does not completely fix $e_2$ and $e_3$, and *in cases A(ii) and B(ii), we will choose $e_2$ to be parallel to a HO KVF $\xi$* (see the Appendix). By theorem 2.1, we can redefine the other KVF $\eta$ to be parallel to $e_3$.

With this choice of frame, it follows from theorem 3.1 of paper I, and the Appendix, that $R^*_{\alpha\beta}$ and $C^*_{\alpha\beta}$ have the following forms.

*Classes A(i) and B(i):*

$$(R^*_{\alpha\beta}) = \begin{pmatrix} R^*_{11} & 0 & 0 \\ 0 & R^*_{22} & R^*_{23} \\ 0 & R^*_{23} & R^*_{33} \end{pmatrix}, \qquad (C^*_{\alpha\beta}) = \begin{pmatrix} C^*_{11} & 0 & 0 \\ 0 & C^*_{22} & C^*_{23} \\ 0 & C^*_{23} & C^*_{33} \end{pmatrix}. \tag{2.1}$$

*Classes A(ii) and B(ii):*

$$(R^*_{\alpha\beta}) = \mathrm{diag}(R^*_{11}, R^*_{22}, R^*_{33}), \qquad (C^*_{\alpha\beta}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & C^*_{23} \\ 0 & C^*_{23} & 0 \end{pmatrix}. \tag{2.2}$$

The form of the shear tensor for the four classes is as follows.

$$\textit{Class}\ \mathrm{A(i)}: \qquad (\sigma_{\alpha\beta})\ \text{arbitrary} \tag{2.3}$$

$$\text{Class A(ii):} \qquad (\sigma_{\alpha\beta}) = \begin{pmatrix} \sigma_{11} & 0 & \sigma_{13} \\ 0 & \sigma_{22} & 0 \\ \sigma_{13} & 0 & \sigma_{33} \end{pmatrix}, \tag{2.4}$$

$$\text{Class B(i):} \qquad (\sigma_{\alpha}\beta) = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & \sigma_{23} \\ 0 & \sigma_{23} & \sigma_{33} \end{pmatrix}. \tag{2.5}$$

$$\text{Class B(ii):} \qquad (\sigma_{\alpha\beta}) = \text{diag}(\sigma_{11}, \sigma_{22}, \sigma_{33}). \tag{2.6}$$

These results follows from theorem 3.1 of paper I and the Appendix.

*Remark.* The relationship of solutions which admit an Abelian $G_2$ to the classification scheme of paper I has been given in that paper (see § 8): the intrinsic geometry is class I, and the extrinsic geometry is class $A_3$ if the group does not act orthogonally transitively, and class $B_3$ if it does. The use of the letters A and B to label the solutions in this section is thus consistent. Solutions of class A(i) (respectively B(i)) are not distinguished from solutions of type A(ii) (respectively B(ii)) within the classification scheme of paper I. All four classes, however, can be distinguished by considering the relationship between the eigenframes of $R^*_{\alpha\beta}$ and $\sigma_{\alpha\beta}$, as follows.

A(i): The Ricci and shear eigenframes are not aligned in any way.

A(ii): $R^*_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ have a common eigenvector, which is defined by the HO KVF.

B(i): $R^*_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ have a common eigenvector, which is orthogonal to the group orbits and the fluid flow vector.

B(ii): $R^*_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ have a common eigenframe.

We note that solutions of class B(ii) are particularly simple. The shear and spatial Ricci tensors have a common eigenframe, which is Fermi propagated (see equations (A2) and (A5)). In addition, if $C^*_{\alpha\beta} \neq 0$ and $R^*_{\alpha\beta}$ has a uniquely defined eigenframe (up to reflections), then the eigenframes of these tensors are related by a rotation through $\pi/4$ radians in the two-space spannned by the KVFS.

## 3. Solutions which admit an Abelian $G_2$: local coordinates

In this section, we describe the construction of canonical coordinates, based on two commuting KVFS $\xi$ and $\eta$, and on a HO time-like vector field $u$ of unit length, which is orthogonal to $\xi$ and $\eta$. This vector field will be interpreted as the velocity field of the perfect fluid, but the construction of the coordinates does not depend on this, or on the field equations. Note that by the lemma† in the Appendix of paper I, $u$ has zero Lie derivative with respect to the KVFS, ie.

$$[u, \xi] = 0 = [u, \eta], \tag{3.1}$$

in terms of the Lie bracket.

*Theorem* 3.1. Suppose that the space–time admits two space-like commuting KVFS $\xi$ and $\eta$. Let $u$ be a HO time-like vector field of unit length, which is orthogonal to $\xi$ and

† In the statement of this lemma, the requirement that the vector field $\eta$ be non-null and of constant length was inadvertently omitted.

$\eta$. Then there exist local coordinates $(t, x, y, z)$ such that

$$u = e^{-k}\, \partial/\partial t, \qquad u_i\, dx^i = e^k\, dt, \tag{3.2a}$$

$$\xi = \partial/\partial y, \qquad \eta = \partial/\partial z, \tag{3.2b}$$

$$ds^2 = -e^{2k(t,x)}\, dt^2 + g_{\alpha\beta}(t, x)\, dx^\alpha\, dx^\beta, \tag{3.2c}$$

where $\alpha, \beta = 1, 2, 3$ and $(x^\alpha) = (x, y, z)$.

*Proof.* Since $u$ is HO, we can choose $t$ such that

$$u_i\, dx^i = e^k\, dt.$$

We then choose the spatial coordinates so that $x^\alpha{}_{,i} u^i = 0$. Then (3.2$a$) will hold. Since $\xi$ and $\eta$ are orthogonal to $u$ and satisfy (3.1), it follows that

$$\xi = \xi^\alpha\, \partial/\partial x^\alpha, \qquad \eta = \eta^\alpha\, \partial/\partial x^\alpha,$$

where $\xi^\alpha$ and $\eta^\alpha$ are independent of $t$. We can thus regard $\xi$ and $\eta$ as vector fields in a three-dimensional manifold. Since they commute, we can introduce spatial coordinates so that (3.2$b$) holds. The coordinate dependence of $k$ and the $g_{\alpha\beta}$ follows from (3.1) and the Killing equations.

*Remark.* The remaining freedom in choice of the coordinates is

$$\begin{aligned} t' &= f^0(t), & x' &= f^1(x), \\ y' &= y + f^2(x), & z' &= z + f^3(x). \end{aligned} \tag{3.3}$$

The $x$ coordinate can be interpreted geometrically as follows. The vector fields $u$, $\xi$ and $\eta$ commute pairwise and hence are tangent to a family of hypersurfaces. Thus the unit vector field $v$ which is orthogonal to $u$, $\xi$ and $\eta$ is HO. From equations (3.2$a$, $b$) and the orthogonality properties it follows that

$$v_a\, dx^a = e^h\, dx, \tag{3.4}$$

where $e^{2h} = 1/g^{xx}$. In other words, $x$ defines the hypersurfaces spanned by $u$, $\xi$ and $\eta$. The orthonormal frame used in § 2 and in the Appendix consists of $e_{(0)} = u$, $e_{(1)} = v$ and two suitable linear combinations of $\xi$ and $\eta$. In view of (3.4) it is convenient to complete the square in the line-element (3.2$a$), and write

$$ds^2 = -e^{2k}\, dt^2 + e^{2h}\, dx^2 + r[f(dy + w_1\, dz + w_2\, dx)^2 + f^{-1}(dz + w_3\, dx)^2], \tag{3.5}$$

where $k, h, r, f$ and the $w_\alpha$ are functions of $t$ and $x$. The orthonormal frame which is used in §2 and in the Appendix is defined by

$$\begin{aligned} w^{(0)} &= e^k\, dt, & w^{(1)} &= e^h\, dx, \\ w^{(2)} &= (rf)^{1/2}(dy + w_1\, dz + w_2\, dx), & w^{(3)} &= (r/f)^{1/2}(dz + w_3\, dx). \end{aligned} \tag{3.6}$$

The coordinates constructed in theorem 3.1 are canonical coordinates for space-times of class A(i). The following theorem describes the specialisation that arises in connection with the classification of § 2.

*Theorem* 3.2. For space–times of classes A(ii), B(i) and B(ii), local coordinates can be chosen so that $u$, $\xi$ and $\eta$ are given by (3.2*a*, *b*), and the line element by (3.5), with

$$(1)\ w_1 = w_2 = 0 \quad \text{or} \quad w_1 = w_3 = 0, \quad \text{in class A(ii)}, \tag{3.7$a$}$$

$$(2)\ w_2 = w_3 = 0, \quad \text{in class B(i)}, \tag{3.7$b$}$$

$$(3)\ w_1 = w_2 = w_3 = 0, \quad \text{in class B(ii)}. \tag{3.7$c$}$$

*Proof.* For classes B(i) and B(ii), the existence of these coordinates is well known (although usually the group orbits are time-like; see for example Carter (1973, p 165)). For class A(ii), an outline of the proof is given in the Appendix.

*Remarks.* (1) The metric functions $r, f$ and $w_1$ are in fact scalars which can be expressed in terms of the KVFS, using (3.2*b*), as follows†:

$$r^2 = (\xi \cdot \xi)(\eta \cdot \eta) - (\xi \cdot \eta)^2, \qquad rf = \xi \cdot \xi, \qquad rfw_1 = \xi \cdot \eta. \tag{3.8}$$

The scalar $r$ provides a useful criterion for classifying space–times which admit an Abelian $G_2$. The criterion is the nature of the hypersurfaces $r$ = constant (assuming $r$ is not identically constant). The three cases are distinguished by whether the scalar

$$\sum = r_{,a} r_{,b} g^{ab} \tag{3.9}$$

is positive (space-like hypersurface), negative (time-like hypersurface) or zero (null hypersurface).

(2) For vacuum solutions one is not tied to a particular vector field $u$, and so one has the freedom to set $h = k$ in equation (3.5).

## 4. Exact solutions with an Abelian $G_2$

In this section we survey the known exact solutions of the Einstein field equations which fall within the framework of §§ 2 and 3, i.e. which admit an Abelian $G_2$ whose orbits are orthogonal to the fluid velocity (assumed irrotational). We assume that condition $A_1$ of the Introduction is satisfied. Before giving details, we note that surprisingly few solutions of this type, with a perfect fluid source, are known. (See for example Kramer *et al* (1980, ch 13, §§ 5, 6, ch 15 and ch 20, § 5.)) The author is in fact aware of four sources of such solutions.

(1) Locally rotationally symmetric (LRS) solutions in which a $G_3$ acts on 2D space-like surfaces of zero intrinsic curvature (class II, with $K = 0$, in Stewart and Ellis (1968)). The general solution for the case of dust ($p = 0$) is known (see for example, Kramer *et al* 1980, ch 13, § 5, Eardley *et al* 1972)‡.

(2) The Szekeres (1975) dust solutions ($p = 0$) and their generalisation with pressure (Szafron and Wainwright 1977, Szafron 1977), These solutions in general admit no KVFS, but by specialising the metric in an obvious way, one can obtain two commuting KVFS.

† $\xi \cdot \xi$ denotes $g_{ij}\xi^i\xi^j$ etc.

‡ Note that the well known Tolman–Bondi spatially inhomogeneous dust solutions (Tolman 1934, Bondi 1947), which are LRS, do not arise, since they are spherically symmetric, and hence do not admit an Abelian $G_2$.

(3) Solutions with $p = \mu$ (i.e. stiff matter) can be derived from vacuum solutions using the algorithm of Wainwright *et al* (1979). This paper gives some references to earlier work. This algorithm has also been derived independently, and in a slightly different form, by Belinskii (1979).

(4) The author (Wainwright and Goode 1980) has recently discovered a family of solutions, in which an equation of state of the form $p = \gamma\mu$, $0 < \gamma < 1$, is possible.

### 4.1. *Solutions of class B(ii)*

All the solutions mentioned in the preceding survey, with the exception of some of the $p = \mu$ solutions, are of the simplest type, namely class B(ii). The properties of the LRS solutions, as regards intrinsic and extrinsic geometry, were given in paper I. We now give some examples of Szekeres solutions of class B(ii). A simple subclass of the Szekeres dust solutions can be obtained from Szafron and Wainwright (1977), by setting $q = 1$ (see also Bonnor and Tomimura 1976). One obtains

$$ds^2 = -dt^2 + t^{4/3}(dx^2 + dy^2 + Z^2\, dz^2), \tag{4.1}$$

where

$$Z = A + F + Bt^{-1} - \tfrac{9}{10}Ct^{2/3},$$

with

$$F = ax + by - \tfrac{1}{2}C(x^2 + y^2).$$

Here $A$, $B$, $C$, $a$ and $b$ are arbitrary functions of $z$. The coordinates are comoving, and the matter density is

$$8\pi\mu = 4(A+F)/3t^2 Z.$$

Three possible specialisations which give rise to two commuting, HO, mutually orthogonal KVFs are given below, with the KVFs as indicated:

(1) $a = b = C = 0$: $\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y}$;

(2) $b = C = 0$, $a, A, B = \text{constant}$, $a \neq 0$: $\dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}$;

(3) $a = b = 0$, $A, B, C = \text{constant}$, $C \neq 0$: $y\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}$.

By making suitable identifications (and, in case (3), a coordinate transformation $x = r\cos\theta$, $y = r\sin\theta$), one can relate the line element (4.1) to the canonical line element (3.5), subject to (3.7*c*). As pointed out by Szafron and Collins (1979), the Szekeres solutions are quite special as regards intrinsic and extrinsic geometry (see § 4 of this paper for details). Solution (1) above is LRS and has been studied by Szafron and Wainwright (1977).

Solutions of class B(ii), with $p = \mu$, have been given by Wainwright *et al* (1979, examples 1(*a*) and 2(*a*)). It has been shown by McIntosh (1978) that these solutions, which are not spatially homogeneous, are in fact spatially self-similar, i.e. they admit a three-parameter group of homothetic motions, generated by the two KVFs and a homothetic vector field. They are more general than the above specialisations of the

Szekeres solutions in that *the slices are not conformally flat in general*, and $R^*_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ do not have a repeated eigenvalue.

A new family of solutions in class B(ii) has recently been found (Wainwright and Goode 1980). These solutions are not spatially self-similar and include a subset with equation of state $p = \gamma\mu$, $0 < \gamma < 1$. Full details concerning the extrinsic and intrinsic geometry are given in Wainwright and Goode (1980).

We note that in all examples except the preceding Szekeres solutions, cases (2) and (3), the scalar $\Sigma$, as defined by (3.9), is negative. In the Szekeres solutions, the expression for $\Sigma$ is complicated and it probably changes sign over the space–time. The reason that $\Sigma < 0$ in most of the solutions is that this condition was imposed during the derivation. This choice was made because consideration of spatially homogeneous cosmologies with two commuting KVFs suggested that $\Sigma < 0$ was the appropriate choice for cosmological solutions (see also Liang 1976), and this was our main interest.

### 4.2. *Solutions of class B(i)*

The only perfect fluid solutions in this class of which the author is aware are solutions with equation of state $p = \mu$, derived using the algorithm of Wainwright *et al* (1979) and Belinskii (1979). In principle, many such solutions could be derived using this algorithm, since the required vacuum solutions could be generated using the Bäcklund transformation technique (see for example Harrison (1980)) or the inverse scattering problem technique of Belinskii and Zakharov (1978). Two classes of physically interesting solutions have been written down explicitly. These are the gravitational wave solutions of Wainwright and Marshman (1979), and the one-soliton cosmological wave solutions of Belinskii (1979). We refer to Wainwright (1979) for a detailed analysis of one of the gravitational wave solutions. In addition, some of the solutions given by Wainwright *et al* (1979, examples 1(*b*) and 2(*b*)) are in class B(i). These particular solutions are in fact spatially self-similar (McIntosh 1978). Finally we note that the scalar $\Sigma$, as defined by (3.9), changes sign in the Belinskii (1979) solutions, while it is negative in the remaining solutions.

### 4.3. *Solutions of classes A(i) and A(ii)*

The author is not aware of any solutions in these classes. It should prove possible to derive solutions of class A(ii).

Finally, for completeness, we relate some of the well known spatially inhomogeneous *vacuum* solutions to the classification of § 2. For example, the cylindrical gravitational wave solutions of Einstein and Rosen (see, for example, Weber and Wheeler (1957)), the Kahn and Penrose (1971) and Szekeres (1972) colliding plane wave solutions, and the Gowdy (1971) universes, which describe gravitational waves propagating in a closed universe, all belong to class B(ii).

## 5. Algebraically special solutions

The known, algebraically special, spatially inhomogeneous, perfect fluid solutions can be grouped into three classes, which are distinguished as follows. In an algebraically special space–time, there is a congruence of null curves whose tangent vector field defines, at each point, a repeated principal null direction of the Weyl tensor (see Kramer *et al* (1980, pp 61, 90) for this terminology). This congruence is called a *repeated principal null congruence* (PNC). the defining properties of the three classes are:

(1) the repeated PNC is geodesic and shear-free;

(2) the repeated PNC is geodesic and has non-zero shear;

(3) the repeated PNC is non-geodesic.

In each class of examples to be presented below, the repeated PNC has zero twist but non-zero expansion. In addition, the solutions admit no KVFs in general. For completeness, we note that all LRS perfect fluid solutions satisfy (1) (Wainwright 1970).

### *5.1. Geodesic and shear-free PNC*

A simple class of cosmological solutions of this type has the following line element (Wainwright 1974):

$$ds^2 = r(dx^2 + dy^2) - 2\, dv\, dr + U\, dv^2,$$

where $U = U(x, y, v)$ satisfies the partial differential equation

$$U_{xx} + U_{yy} + U_v = 0.$$

The fluid velocity, density and pressure are given by

$$u_a\, dx^a = U^{-1/2}\, dr,$$

and

$$8\pi\mu = 8\pi p = U/(4r^2).$$

The repeated PNC is defined by the vector field

$$l = \partial/\partial r,$$

and is geodesic, shear-free and twist-free (Wainwright 1974). Indeed, the derivation of these solutions was based on the assumption that such a repeated PNC existed.

The coordinate system is unusual for a cosmological model, in that it consists of three space-like coordinate $x$, $y$, $v$ (note that we require $U > 0$ on physical grounds), and one null coordinate $r$. In addition, the hypersurfaces orthogonal to the fluid flow are defined by $r =$ constant. These coordinates are certainly not comoving, and we have been unable to find comoving coordinates except for very simple choices of $U$. A more general class of solutions has been given by Wainwright (1974), but the subclass presented here is general enough to illustrate the properties of this type of solution.

An orthonormal frame, with $e_{(0)} = u$, is defined by the following one-forms:

$$w^{(0)} = f^{-1}\, dr, \qquad w^{(1)} = r^{1/2}\, dx, \qquad w^{(2)} = r^{1/2}\, dy, \qquad w^{(3)} = -f^{-1}\, dr + f\, dv,$$

where

$$f = \sqrt{U}.$$

The dual basis of vector fields is given by

$$e_{(0)} = f\frac{\partial}{\partial r} + \frac{1}{f}\frac{\partial}{\partial v}, \qquad e_{(1)} = \frac{1}{f}\frac{\partial}{\partial v},$$

$$e_{(2)} = r^{-1/2}\, \partial/\partial x, \qquad e_{(3)} = r^{-1/2}\, \partial/\partial y.$$

Relative to this frame, the non-zero intrinsic components are

$$R^*_{11} = -(rf)^{-1} f_{xx}, \qquad R^*_{12} = -(rf)^{-1} f_{xy},$$
$$R^*_{22} = -(rf)^{-1} f_{yy}, \qquad R^*_{33} = R^*_{11} + R^*_{22},$$
$$C^*_{11} = -C^*_{22} = -(rf)^{-1} B_v, \qquad C^*_{12} = (rf)^{-1} A_v,$$
$$C^*_{13} = r^{-3/2}(-A_y + B_x), \qquad C^*_{23} = r^{-3/2}(-A_x - B_y),$$

where

$$A = (f_{xx} - f_{yy})/f, \qquad B = 2f_{xy}/f.$$

The non-zero extrinsic components are

$$\dot{u}_1 = -r^{-1/2} f_x/f, \qquad \dot{u}_2 = -r^{-1/2} f_y/f, \qquad \dot{u}_3 = -f_v/f^2, \qquad \theta = r^{-1}f - f_v/f^2,$$
$$\sigma_{11} = \sigma_{22} = W, \qquad \sigma_{33} = -2W, \qquad \sigma_{13} = -r^{-1/2} f_x/f, \qquad \sigma_{23} = -r^{-1/2} f_y/f,$$

where

$$W = \tfrac{1}{6}(r^{-1}f + 2f_v/f^2).$$

Inspection of the preceding formulae shows that *these solutions are of the most general type as regards the classification scheme of paper I*, i.e. class A, extrinsically and class I intrinsically, unless extra restrictions are imposed, e.g. $\partial U/\partial y = 0$. These and subsequent calculations in this section were performed using a library of programs (Wainwright 1978), written in the symbolic algebra language CAMAL (Fitch 1976).

### *5.2. Geodesic* PNC *with non-zero shear*

We consider a subclass of the general, type-N, irrotational perfect fluid solutions (Oleson 1971). The line element is given (see Oleson (1971), subject to an obvious coordinate transformation and relabelling) by

$$\mathrm{d}s^2 = r^3\left(\mathrm{d}x - \frac{1}{r}H_x\,\mathrm{d}v\right)^2 + r(\mathrm{d}y + rH_y\,\mathrm{d}v)^2 - 2rH\,\mathrm{d}r\,\mathrm{d}v + (rf^2 + r^3k^2)H^2\,\mathrm{d}v^2,$$

where $f = f(v)$, $k$ = constant and $H = H(v, x, y)$ are subject to

$$H_{xx} + f^2 H = 0, \qquad H_{yy} + k^2 H = 0.$$

The fluid velocity, density and pressure are given by

$$u_a\,\mathrm{d}x^a = \frac{r}{(rf^2 + r^3k^2)^{1/2}}\,\mathrm{d}r,$$
$$8\pi p = (3/4r^3)(f^2 - 7r^2k^2), \qquad 8\pi(\mu - p) = 12k^2/r.$$

The repeated PNC is defined by the vector field

$$l = \partial/\partial r,$$

and is geodesic, but has non-zero shear (Oleson 1971). These solutions were derived by assuming that the Weyl tensor is of Petrov type N.

The coordinate system is of the same nature as in the preceding subsection, i.e. the coordinates are not comoving. We note that when $f'(v) = 0$, this solution becomes a Friedman–Robertson–Walker solution, although this is not obvious by inspection.

An orthonormal frame which is adapted to the fluid flow is defined by

$$w^{(0)} = (r^{1/2}/Z)\,\mathrm{d}r, \qquad w^{(1)} = r^{3/2}[\mathrm{d}x - (1/r)H_x\,\mathrm{d}v],$$
$$w^{(2)} = r^{1/2}(\mathrm{d}y + rH_y\,\mathrm{d}v), \qquad w^{(3)} = r^{1/2}ZH\,\mathrm{d}v - (r^{1/2}/Z)\,\mathrm{d}r,$$

where

$$Z = (f^2 + k^2r^2)^{1/2}.$$

The dual basis of vector fields is given by

$$e_{(0)} = r^{-1/2}Z\,\partial/\partial r + e_{(3)},$$
$$e_{(1)} = r^{-3/2}\,\partial/\partial x, \qquad e_{(2)} = r^{-1/2}\,\partial/\partial y,$$
$$e_{(3)} = (r^{-1/2}/ZH)(\partial/\partial v + r^{-1}H_x\,\partial/\partial x - rH_y\,\partial/\partial y).$$

Relative to this frame, the non-zero extrinsic quantities are as follows:

$$\dot{u}_3 = -\chi/r^{1/2}Z, \qquad \sigma_{11} = \sigma_{22} = \chi/3r^{1/2}Z, \qquad \sigma_{33} = -2\sigma_{11},$$
$$\theta = \frac{3f^2}{2r^{3/2}Z^3} - \frac{\chi}{r^{1/2}Z} + \frac{9k^2r^{1/2}}{Z^{3/2}},$$

where

$$\chi = ff'/Z^2H.$$

The non-zero components of $R^*_{\alpha\beta}$ and $C^*_{\alpha\beta}$ are given by

$$R^*_{11} = \frac{\chi}{r^2Z^2}(f^2 + 2k^2r^2) + \frac{2k^2f^2}{rZ^2}, \qquad R^*_{22} = \frac{\chi k^2}{Z^2} + \frac{2k^2f^2}{rZ^2},$$
$$R^*_{33} = \frac{\chi}{r^2Z^2}(f^2 + 3k^2r^2) + \frac{2k^2f^2}{rZ^2},$$
$$C^*_{11} = \frac{2\chi}{r^{5/2}Z}\left(\frac{H_{xy}}{H}\right), \qquad C^*_{22} = -C^*_{11}, \qquad C^*_{33} = 0,$$
$$C^*_{12} = \frac{\chi}{r^{3/2}Z}\left[\frac{1}{r^2}\left(\frac{H_x}{H}\right)^2 - \left(\frac{H_y}{H}\right)^2 - 2k^2 - \frac{1}{H}\frac{\chi_v}{\chi}\right],$$
$$C^*_{13} = -\frac{\chi}{r^{5/2}}\left(\frac{H_y}{H}\right), \qquad C^*_{23} = -\frac{\chi}{r^{7/2}}\frac{H_x}{H}.$$

Inspection of the preceding formulae reveals that the solutions are extrinsically of class $B_1$, and intrinsically of class I, unless extra conditions are imposed. Note that the frame in use is both a shear eigenframe and an eigenframe of $R^*_{\alpha\beta}$.

### *5.3. Non-geodesic repeated* PNC

The only solutions of this type of which we are aware are the Szekeres solutions and their generalisations (see § 1 for references). These solutions in fact admit two repeated PNCs which are non-geodesic, and are hence of Petrov type D (Wainwright 1977) although, as pointed out in the Introduction, this property was not used to derive the solutions. Their intrinsic and extrinsic geometry has been studied in detail by Szafron and Collins (1979). The acceleration vector is zero, and the spatial gradient of the expansion scalar is not a shear eigenvector. The extrinsic geometry is thus of class $C_1$. As regards intrinsic geometry, these solutions are of class V, since the Cotton–York tensor is zero. In addition, $R^*_{\alpha\beta}$ and $\sigma_{\alpha\beta}$ have a common eigenframe and both tensors have a repeated eigenvalue.

## 6. Conclusion

In this paper we have surveyed and classified *all* known exact solutions of the EFEs whose source is a perfect fluid with *non-zero expansion*, and such that the *maximal group of local isometries has dimension* $\leq 2$. As pointed out in the Introduction, in all these solutions the perfect fluid has zero vorticity. We have not given complete details on the known LRS solutions, however, since this is available elsewhere (see Kramer *et al* (1980) for explicit solutions, and Wainwright (1979) for the properties of their extrinsic and intrinsic geometry). For reasons given in the Introduction, it was sufficient to restrict our considerations to solutions which admit an *Abelian* $G_2$ (§§ 2, 3, 4) and to solutions whose Weyl tensor is *algebraically special* (§ 5).

In order to give a clear picture of the current situation, it should be stressed that the known exact solutions which satisfy $A_1$ and $A_2$ are quite specialised, despite the fact that some of the solutions admit no KVFs. We pointed out in the Introduction that all the known solutions satisfy restrictions $R_1$ or $R_2$, or both. We list here two other restrictions which are often satisfied:

$R_3$: the slices orthogonal to the fluid flow are conformally flat, i.e. $C^*_{\alpha\beta} = 0$;

$R_4$: the equation of state (if it exists) is extreme, i.e. $p = 0$ or $p = \mu$.

We stress that none of the conditions $R_1$–$R_4$ is desirable, and the extent to which they are satisfied gives a measure of the specialised nature of solutions. The table below summarises whether or not conditions $R_1$–$R_4$ are satisfied by the spatially inhomogeneous solutions which have been discussed in this paper.

| Class of solutions | $R_1$ | $R_2$ | $R_3$ | $R_4$ |
|---|---|---|---|---|
| 1. Spherically or plane symmetric dust | YES | YES | YES | YES |
| 2. Szekeres (1975) | NO | YES | YES | YES |
| 3. Type-N solutions | NO | YES | NO | YES |
| 4. Wainwright (1974) | NO | YES | NO | YES |
| 5. $p = \mu$ solutions | YES | NO | NO | YES |
| 6. Wainwright and Goode (1980) | YES | NO | YES | NO |

Another problem that is encountered with these solutions is that often the pressure and density of the fluid become unbounded on *each* slice, unless the slices and hence the space–time are artificially restricted. This problem can be avoided in classes 1, 2, 5 and 6.

A positive feature to look for in spatially inhomogeneous solutions is the presence of essential arbitrary functions, since this increases the generality of the solutions. Classes 1 and 2 above contain one or more essential arbitrary functions of one space-like coordinate, while class 3 and certain solutions in class 5 (see Wainwright and Marshman 1979) contain an essential arbitrary function which is constant on null hypersurfaces.

One can also study whether a spatially inhomogeneous solution approximates a Friedman–Robertson–Walker solution at late times. To date, in all solutions which have been found to have this property, the fluid has zero acceleration (which implies $p=0$, if one demands an equation of state $p=p(\mu)$, and spatial inhomogeneity). We refer to Bonnor (1974) for class 1 solutions, in the preceding table, and to Bonnor and Tomimura (1976) and Szafron and Wainwright (1977) for class 3 solutions†.

The behaviour of spatially inhomogeneous solutions near the initial singularity is also of interest. It has been conjectured by Penrose (see for example Hawking and Israel (1979, pp 630–1)), and Barrow and Matzner (1977) that near the initial singularity the universe should in some sense approach homogeneity and isotropy. These spatially inhomogeneous solutions are currently being studied by the author from this point of view.

In this paper, we have studied the known non-rotating spatially inhomogeneous solutions from the point of view of the intrinsic and extrinsic geometry of the hypersurfaces orthogonal to the fluid flow. While this approach provides a common framework for stydying such solutions, it has as yet not led to the discovery of any new models. The usefulness of this approach as a technique for finding solutions, first suggested by Collins (1979), would be established, for example, if it were to lead to the discovery of solutions which satisfied neither restriction $R_1$ or $R_2$.

We conclude by mentioning some other possibilities for deriving new spatially inhomogeneous solutions. As regards solutions with a two-parameter group of local isometries, we could consider a group which does not act orthogonally transitively, or a group which is non-Abelian. A more profitable approach would probably be to assume the existence of *one* hypersurface-orthogonal KVF, which is orthogonal to the fluid flow, together with some restriction on the intrinsic or extrinsic geometry of the slices. Investigations in these directions are continuing.

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## Appendix

*Proof of theorem 2.1.* As described in § 2, following the statement of theorem 2.1, we select a unit time-like HO vector field $u$, which is orthogonal to the group orbits. As in the proof of theorem 3.1 in paper I, we can choose an orthonormal frame $\{e_a\}$, with $e_2$, $e_3$ tangent to the group orbits, and $e_0=u$, so that

$$[e_a, \xi]=0=[e_a, \eta], \tag{A1}$$

where $\xi$, $\eta$ are two Killing vector fields, which generate the group. Since the group is Abelian, we have the following restrictions on the connection coefficients:

$$\begin{aligned} &a_2=a_3=0, \qquad n_{11}=n_{12}=n_{13}=0, \\ &\sigma_{12}+\Omega_3=\sigma_{13}-\Omega_2=0, \qquad \dot{u}_2=\dot{u}_3=0. \end{aligned} \tag{A2}$$

† The discussions given to date have been based on particular local coordinates, and in the author's opinion, the concepts involved have not been adequately defined.

There is still the following freedom in choice of the frame:

$$\tilde{e}_2 = \cos\psi\, e_2 + \sin\psi\, e_3, \qquad \tilde{e}_3 = -\sin\psi\, e_2 + \cos\psi\, e_3, \tag{A3}$$

where $\psi$ is constant on the group orbits, i.e. $e_2(\psi) = 0$, $e_3(\psi) = 0$. We can write

$$\xi = Ae_2 + Be_3, \qquad \eta = Ce_2 + De_3,$$

and since the group is Abelian, it follows that the functions $A, B, C, D$ are constant on the group orbits. Hence we can use (A3) to set $B = 0$, i.e.

$$\xi = Ae_2, \qquad \eta = Ce_2 + De_3. \tag{A4}$$

It follows from (A1) and the commutators that

$$\sigma_{23} - \Omega_1 = 0, \qquad n_{33} = 0 \tag{A5}$$

We now use the fact that $\xi$ is hypersurface-orthogonal, i.e. that $e_0$, $e_1$ and $e_3$ are tangent to hypersurfaces. This means that the commutators of $e_0$, $e_1$ and $e_3$ are linear combinations of $e_0$, $e_1$ and $e_3$. The commutators thus imply that

$$\sigma_{12} - \Omega_3 = 0, \qquad \dot{\sigma}_{23} + \Omega_1 = 0, \qquad n_{22} = 0.$$

In conjugation with (A2) and (A5) this yields

$$\begin{aligned} &a_2 = a_3 = 0, \qquad \Omega_1 = \Omega_3 = 0, \qquad n_{11} = n_{12} = n_{13} = n_{22} = n_{33} = 0,\\ &\sigma_{12} = \sigma_{23} = 0, \qquad \dot{u}_2 = \dot{u}_3 = 0, \qquad \sigma_{13} - \Omega_2 = 0. \end{aligned} \tag{A6}$$

We now conclude, using theorem 3.1 and equations (A4) and (A5) of paper I, that

$R^*_{\alpha\beta}$ is diagonal

and

$C^*_{23}$ is the only non-zero component of $C^*$.

By making use of (A1), (A4), (A6) and the commutators, we find that

$$e_0(A/C) = 0, \qquad e_1(A/C) = 0,$$

which implies that

$$C = kA, \qquad k = \text{constant}.$$

Hence the vector field

$$\hat{\eta} = \eta - k\xi = De_3$$

is a Killing vector field, which is orthogonal to $\xi$. Inspection of the commutators shows that $e_3$, and hence $\hat{\eta}$, is hypersurface-orthogonal iff $\sigma_{13} + \Omega_2 = 0$. But because of the restrictions on the connection coefficients, this is precisely the condition for the group to act orthogonally transitively.

*Proof of Theorem* 3.2. We briefly indicate how to construct canonical coordinates for class A(ii) space–times. In this case the frame vectors $e_{(0)}$, $e_{(1)}$ and $e_{(2)}$ are hypersurface-orthogonal (see the preceding proof and the Appendix in paper I). Thus there exist coordinates $t$, $x$, $y$, $z$ such that the dual one-forms are given by

$$w^{(a)} = A\,\mathrm{d}t, \qquad w^{(1)} = B\,\mathrm{d}x, \qquad w^{(2)} = C\,\mathrm{d}y, \qquad w^{(3)} = T\,\mathrm{d}t + X\,\mathrm{d}x + Y\,\mathrm{d}y + Z\,\mathrm{d}z,$$

where the coefficients are arbitary functions at this stage. The exterior derivatives of these one-forms are given by

$$\mathrm{d}w^{(a)} = -\tfrac{1}{2}C^a_{bc}w^{(b)} \wedge w^{(c)} \tag{A7}$$

where the $C^a_{bc}$ are the structure coefficients of the orthonormal frame, i.e.

$$[e_{(a)}, e_{(b)}] = C^c_{ab}e_{(c)}$$

(see for example Ryan and Shepley (1975, p 27)). The $C^c_{ab}$ can be read off from the usual expressions for the commutators in the Ellis–MacCallum tetrad formalism (see for example the Appendix in paper I), and are restricted by equations (A6).

Firstly it follows that $A$, $B$ and $C$ are functions of $t$, $x$ only. (As regards $C$, the $y$-dependence factors out and is absorbed into d$y$ by redefining $y$). Secondly, we can use the freedom in the $z$ coordinate to set $Y=0$. The commutators then imply that $Z$, $X$ and $T$ are independent of $y$, and so the $z$ coordinate can be further specialised to set $T=0$. Further use of the comutators and the fact that the structure coefficients are independent of $y$ and $z$ enables one to use the remaining $z$-freedom to eliminate the $z$ dependence in $X$ and $Z$. The frame thus has the form (3.6), with $w_1 = w_2 = 0$.

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